COULOMB'S LAW AND ELECTRIC FIELD INTENSITY

قانون كولوم وشدة المجال الكهربائي

1. The Experimental Law of Coulomb

Coulomb stated that "*The force between two very small objects separated in a vacuum or free space by a distance which is large compared to their size is proportional to the charge on each and inversely proportional to the square of the distance between them*".

قانون كولوم: " القوة بين جسمين صغيرين جدا يفصلهما في الفراغ أو الفضاء الحر مسافة كبيرة بالنسبة لمقاييسها تتناسب طرديا مع الشحنة على كل منهما وتتناسب عكسيا مع مربع المسافة بينهما".

$$F = \frac{Q_1 Q_2}{4\pi\varepsilon_0 R^2}$$

Where:

F: Force in newton (N),

Q_1 and Q_2 are the positive or negative quantities of charge in Coulomb(C),

R: is the separation in meters (m)

ε_0 is called the *permittivity of free space* and has the magnitude, measured in farads per meter (F/m)

$$\varepsilon_0 = \frac{1}{36\pi}10^{-9} = 8.854 \times 10^{-12} \qquad \text{F/m}$$

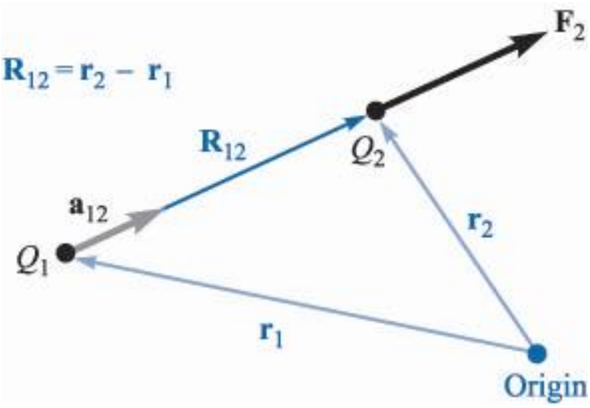
The coulomb is an extremely large unit of charge, for the smallest known quantity of charge is that of the electron (negative) or proton (positive), given in mks units as 1.602×10^{-19} C; hence a negative charge of one coulomb represents about 6×10^{18} electrons.

If point charges Q_1 and Q_2 are located at points having position vector $\mathbf{r_1}$ and $\mathbf{r_2}$, then the vector force $\mathbf{F_{12}}$ on Q_2 duo to Q_1, shown in Figure 2.1, is given by

$$F = \frac{Q_1 Q_2}{4\pi\varepsilon_0 |R|^2}\, \mathbf{a}_{R_{12}}$$

$$\vec{R}_{12} = r_2 - r_1$$

$$\mathbf{a}_{R_{12}} = \frac{\vec{R}_{12}}{|\vec{R}_{12}|}$$



If Q_1 located at (x_0, y_0, z_0) and Q_2 at (x_1, y_1, z_1), then

$$\vec{R}_{12} = (x_1 - x_0)\mathbf{a_x} + (y_1 - y_0)\mathbf{a_y} + (z_1 - z_0)\mathbf{a_z}$$

$$|\vec{R}_{12}| = \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2 + (z_1 - z_0)^2}$$

$$\mathbf{a}_{R_{12}} = \frac{\vec{R}_{12}}{|\vec{R}_{12}|} = \frac{(x_1 - x_0)\mathbf{a_x} + (y_1 - y_0)\mathbf{a_y} + (z_1 - z_0)\mathbf{a_z}}{\sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2 + (z_1 - z_0)^2}}$$

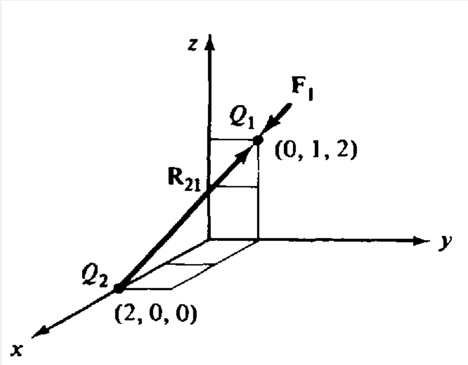
$$F_2 = \frac{Q_1 Q_2}{4\pi\varepsilon_0 |R_{12}|^2}\, \mathbf{a}_{R_{12}}$$

$$= \frac{Q_1 Q_2}{4\pi\varepsilon_0[(x_1 - x_0)^2 + (y_1 - y_0)^2 + (z_1 - z_0)^2]} \frac{(x_1 - x_0)\mathbf{a_x} + (y_1 - y_0)\mathbf{a_y} + (z_1 - z_0)\mathbf{a_z}}{\sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2 + (z_1 - z_0)^2}}$$

$$F_2 = \frac{Q_1 Q_2[(x_1 - x_0)\mathbf{a_x} + (y_1 - y_0)\mathbf{a_y} + (z_1 - z_0)\mathbf{a_z}]}{4\pi\varepsilon_0[(x_1 - x_0)^2 + (y_1 - y_0)^2 + (z_1 - z_0)^2]^{\frac{3}{2}}}$$

Example: Find the force on Q_1 (20μC) duo to charge Q_2 (-300μC), where Q_1 located at (0, 1, 2) and Q_2 at (2, 0, 0)?

Solution:



$$R_{21} = (0-2)a_x + (1-0)a_y + (2-0)a_z$$

$$R_{21} = -2a_x + a_y + 2a_z$$

$$|\vec{R}_{21}| = \sqrt{2^2+1^2+2^2} = 3$$

$$F = \frac{Q_1 Q_2}{4\pi\varepsilon_o|R_{21}|^2}\, a_{R_{21}} = \frac{20\times10^{-6}\times(-300)\times10^{-6}}{4\pi\times8.85\times10^{-12}|3|^2}\times\frac{-2a_x+a_y+2a_z}{3}$$

$$F = 4a_x - 2a_y - 4a_z$$

$$|F| = \sqrt{4^2+2^2+4^2} = 6\ N$$

2. The Electric Field Intensity (E) (شدة المجال الكهربائي)

If we now consider one charge fixed in position, say Q_1, and move a second charge slowly around, we note that there exists everywhere a force on this second charge; in other words, this second charge is displaying the existence of a force field. Call this second charge a test charge Qt. The force on it is given by Coulomb's law,

$$F_t = \frac{Q_1 Q_t}{4\pi\varepsilon_0|R_{1t}|^2}\,\mathbf{a}_{R_{1t}}$$

Writing this force as a force per unit charge gives

$$\frac{F_t}{Q_t} = \frac{Q_1}{4\pi\varepsilon_0|R_{1t}|^2}\,\mathbf{a}_{R_{1t}} \qquad (*)$$

The quantity on the right side of (*) is a function only of Q_2 and the directed line segment from Q_2 to the position of the test charge. This describes a vector field and is called the ***electric field intensity***.

Using a capital letter E for electric field intensity, we have finally

$$E = \frac{F_t}{Q_t}$$

$$\vec{E} = \frac{Q_1}{4\pi\varepsilon_0 |R|^2}\ \mathbf{a}_R$$

Where $\vec{E}$ is electric field intensity measured in v/m

Let us arbitrarily locate Q_1 at the center of a spherical coordinate system. The unit vector $\mathbf{a}_R$ then becomes the radial unit vector $\mathbf{a}_r$, and R is r. Hence

$$\vec{E} = \frac{Q_1}{4\pi\varepsilon_0 |r|^2}\ \mathbf{a}_r$$

Example: Find the electric field intensity (E) at (0, 2, 3) due to a point charge Q (0.4μC) located at (2, 0, 4)?

Solution:

$$R = (0-2)\mathrm{a}_x + (2-0)\mathrm{a}_y + (3-4)\mathrm{a}_z$$

$$R = -2\mathrm{a}_x + 2\mathrm{a}_y - \mathrm{a}_z$$

$$|R| = \sqrt{2^2 + 2^2 + 1^2} = 3$$

$$E = \frac{Q}{4\pi\varepsilon_o |R|^2}\ \mathrm{a}_R = \frac{0.4\times 10^{-6}}{4\pi \times 8.85\times 10^{-12} |3|^2} \times \frac{-2\mathrm{a}_x + 2\mathrm{a}_y - \mathrm{a}_z}{3}$$

$$E = -266.4\ \mathrm{a}_x + 266.4\ \mathrm{a}_y - 133.2\ \mathrm{a}_z$$

$$|E| = \sqrt{266.4^2 + 266.4^2 + 133.2^2} = 399.6\ \ V/m$$

3. ***Field of n Point Charge***

Since the coulomb forces are linear, the electric field intensity due to n point charges, Q_1 at r_1 , Q_2 at r_2,and Q_n at r_n is the sum of the forces on Q_t caused by Q_1 and Q_2 acting alone, or

$$\vec{E} = \vec{E}_1 + \vec{E}_2 + \cdots + \vec{E}_n$$

$$= \frac{Q_1}{4\pi\varepsilon_0|R_1|^2}\ \mathbf{a}_{R_1} + \frac{Q_2}{4\pi\varepsilon_0|R_2|^2}\ \mathbf{a}_{R_2} + \cdots + \frac{Q_n}{4\pi\varepsilon_0|R_n|^2}\ \mathbf{a}_{R_n}$$

Example: A charge of -0.3μC is located at (25, -30, 15) (in cm), and a second charge of 0.5μC is at (-10, 8, 12) cm. Find E at: (a) the origin; (b) (15, 20, 50) cm

Solution:

$(a) \quad E = E_1 + E_2$

$$E_1 = \frac{Q_1}{4\pi\varepsilon_0|R_1|^2}\mathrm{a}_{R1}$$

$the\ point\ must\ be\ in\ meter\ (25, -30, 15) in\ Cm = (0.25, -0.3, 0.15) in\ m$

$(-10\ , 8, 12) in\ Cm = (-0.1, 0.08, 0.12) in\ m$

$R_1 = -0.25\ \mathrm{a}_x + 0.3\mathrm{a}_y - 0.15\mathrm{a}_z$

$R_1 = \sqrt{0.25^2 + 0.3^2 + 0.15^2} = 0.418$

$$E_1 = \frac{-0.3 \times 10^{-6}}{4\pi \times 8.85 \times 10^{-12} \times (0.418)^2} * \frac{-0.25\ \mathrm{a}_x + 0.3\mathrm{a}_y - 0.15\mathrm{a}_z}{0.418}$$

$E_1 = 9233.77\mathrm{a}_x - 11080.52\mathrm{a}_y + 5540.26\mathrm{a}_z$

$$E_2 = \frac{Q_2}{4\pi\varepsilon_0|R_2|^2}\mathrm{a}_{R_2}$$

$R_2 = 0.1\mathrm{a}_x - 0.08\mathrm{a}_y - 0.12\mathrm{a}_z$

$|R_2| = 0.175$

$$E_2 = \frac{0.5 \times 10^{-6}}{4\pi \times 8.85 \times 10^{-12} \times (0.175)^2} \times \frac{0.1a_x - 0.08a_y - 0.12a_z}{0.175}$$

$$E_2 = 83888.55\ a_x - 67110.8\ a_y - 100666.26\ a_z$$

$$E = E_1 + E_2 = 93122.3\ a_x - 78190.52\ a_y - 95126\ a_z$$

$$\therefore E = (39.12a_x - 78.19a_y - 95.12a_z)\ KV/m$$

(***b***)

$E\ at (15,20,50) Cm? \quad \rightarrow \quad (0.15,0.2,0.5)m$

$$R_1 = (0.15 - 0.25)a_x + (0.2 - (-0.3))a_y + (0.5 - 0.15)a_z$$

$$\therefore R_1 = -0.1a_x + 0.5a_y + 0.35a_z$$

$$|R_1| = 0.618$$

$$R_2 = (0.15 - (-0.1))a_x + (0.2 - 0.08)a_y + (0.5 - 0.12)a_z$$

$$R_2 = 0.25a_x + 0.12a_y + 0.38a_z$$

$$|R_2| = 0.47$$

$$E = E_1 + E_2 = \frac{Q_1}{4\pi\varepsilon_0 R_1}\ a_{R_1} + \frac{Q_2}{4\pi\varepsilon_0 {R_2}^2} a_{R_2}$$

$$E = \frac{10^{-6}}{4\pi\varepsilon_0}\left[\frac{-0.3}{(0.618)^2} * \frac{-0.1a_x + 0.5a_y + 0.35a_z}{0.618} + \frac{0.5}{(0.47)^2} \times \frac{0.25a_x + 0.12a_y + 0.38a_z}{0.47}\right]$$

$$\therefore E = \frac{10^{-6}}{4\pi\varepsilon_0}\left[0.127a_x - 0.635a_y - 0.44a_z + 1.2a_x + 0.577a_y + 1.83a_z\right]$$

$$E = 11.9a_x - 0.52a_y + 12.4a_z\ \ KV/m$$

4. ***Field Due to a Continuous Volume Charge Distribution***

If we now visualize a region of space filled with a great number of charges separated by minute distances, we see that we can replace this distribution of very small particles with a smooth continuous distribution described by a ***volume charge density*** $(\boldsymbol{\rho_v})$ $^{\boldsymbol{C}}/_{\boldsymbol{m^3}}$.

اذا تصورنا منطقة من الفراغ مملوءة بعدد هائل من الشحنات المنفصلة عن بعضها بمسافات صغيرة جدا ، فاننا نستطيع احلال هذا التوزيع لجسيمات صغيرة جدا بتوزيع املس يوصف بكثافة شحنة حجمية

The total charge within some finite volume is obtained by integrating throughout that volume,

$$Q = \int_{vol} \rho_v \, dv$$

$$\vec{E} = \int_{vol} \frac{\rho_v dv}{4\pi\varepsilon_o R^2} \mathbf{a}_R$$

Example: Calculate the total charge within each of the indicate volumes:

(a) $0.1 \le x, y, z \le 0.2$; $\rho_v = \frac{1}{x^3 y^3}$

(b) $0 \le \rho \le 0.1$, $0 \le \emptyset \le \pi$, $2 \le z \le 4$; $\rho_v = \rho^2 z^2 \sin 0.6\emptyset$

(c) Universe ; $\rho_v = {e^{-2r}}/{r^2}$

Solution:

$(\boldsymbol{a})$

$$Q = \int_{vol} \rho_v \, dv = \iiint \frac{1}{x^3 y^3} \, dxdydz$$

$$Q = \iint \frac{1}{y^3} dydz * \left[\int_{0.1}^{0.2} x^{-3} dx\right] = \iint \frac{1}{y^3} dydz * \left[\int_{0.1}^{0.2} x^{-3} dx\right]$$

$$Q = \left[\frac{-1}{2x^2}\right]_{0.1}^{0.2} \left[\frac{-1}{2y^2}\right]_{0.1}^{0.2} [z]_{0.1}^{0.2} = \left[\frac{-1}{2(0.2)^2} + \frac{1}{2(0.1)^2}\right]\left[\frac{-1}{2(0.2)^2} + \frac{1}{2(0.1)^2}\right](0.2 - 0.1)$$

$$Q = 140.6\,C$$

(b)

$$Q = \int \rho_v\, dv = \int_0^{\pi}\int_2^{4}\int_0^{0.1} \rho^2 z^2 \sin(0.6\emptyset)\rho\, d\rho dz d\emptyset$$

$$Q = \int_0^{\pi}\int_2^{4}\int_0^{0.1} \rho^3 z^2 \sin(0.6\emptyset)\, d\rho dz d\emptyset = \int_0^{\pi}\int_2^{4} \left[\frac{\rho^4}{4}\right]_0^{0.1} z^2 \sin(0.6\emptyset)\, dz d\emptyset$$

$$Q = 25 \times 10^{-6} \int_0^{\pi}\int_2^{4} z^2 \sin(0.6\emptyset)\, dz d\emptyset = 25 \times 10^{-6} \int_0^{\pi} \left[\frac{z^3}{3}\right]_2^4 \sin(0.6\emptyset) d\emptyset$$

$$= 25 \times 10^{-6} \left[\frac{4^3}{3} - \frac{2^3}{3}\right] \int_0^{\pi} \sin(0.6\emptyset)\, d\emptyset = 466.66 \left[\frac{-1}{0.6}\cos(0.6\emptyset)\right]_0^{\pi} = 466.66 \times 10^{-6}[2.1816]$$

$$\therefore Q = 1.018\, mC$$

(c)

$$Q = \int \rho_v dv = \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} \frac{e^{-2r}}{r^2} . r^2 \sin\theta\, dr d\theta d\emptyset$$

$$= \left[\frac{-1}{2} e^{-2r}\right]_0^{\infty} [-\cos\theta]_0^{\pi} [\emptyset]_0^{2\pi} = \left[\frac{-1}{2} e^{-\infty} + \frac{1}{2} e^0\right][1 + 1][2\pi]$$

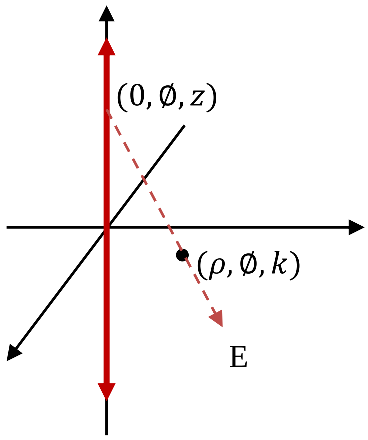
$$Q = \left[0 + \frac{1}{2}\right][2][2\pi] = 6.28C$$

5. ***Field of a Line Charge***

If we now consider a filament like distribution of volume charge density, such as a very fine, sharp beam in a cathode-ray tube or a charged conductor of very small radius, we find it convenient to treat the charge as a line charge of density ρ_L C/m.

اذا اعتبرنا كثافة شحنة حجمية على هيئة توزيع فتيلي، مثل حزمة دقيقة جدا ومركزة في انبونة اشعة الكاثود ، أو موصلا مشحونا اذا كان نصف قطره شديد الصغر، فاننا نجد انة لمن المناسب معاملة الشحنة **كخط شحنة** ذي كثافة ρ_L C/m.

Let us assume a straight line charge extending along the z axis from -∞ to ∞, as shown in Figure. We desire the electric field intensity E at any and every point resulting from a *uniform* line charge density ρ_L



$$Q = \int \rho_L \, dL$$

$$\vec{E} = \int \frac{\rho_L dL}{4\pi\varepsilon_o R^2} \mathbf{a}_R$$

$$dL = dz$$

$$R = (\rho - 0)\mathrm{a}_\rho + (k - z)\mathrm{a}_z = \rho \mathrm{a}_\rho + (k - z)\mathrm{a}_z$$

$$R = \sqrt{\rho^2 + (k - z)^2}$$

$$E = \int \frac{\rho_L dL}{4\pi\varepsilon_0 |R|^2} a_R = \int \frac{\rho_L dz}{4\pi\varepsilon_0 \left(\sqrt{\rho^2 + (k - z)^2}\right)^2} * \frac{\rho \mathrm{a}_\rho + (k - z)\mathrm{a}_z}{\rho^2 + (k - z)^2}$$

$$= \frac{\rho_L}{4\pi\varepsilon_0}\left[\int_{\infty}^{-\infty} \frac{\rho}{[\rho^2 + (k - z)^2]^{3/2}} dz \, \mathrm{a}_\rho + \int_{\infty}^{-\infty} \frac{z_0 - z}{[\rho^2 + (z_0 - z)^2]^{3/2}} dz \, \mathrm{a}_z\right]$$

$$Let \; u = k - z \qquad \therefore du = -dz$$

$$u_2 = k - \infty = -\infty$$

$$u_1 = k + \infty = \infty$$

$$E = \frac{-\rho_L}{4\pi\varepsilon_0}\left[\int_{-\infty}^{\infty} \frac{\rho}{(\rho^2 + u^2)^{3/2}} du \, \mathrm{a}_\rho + \int_{-\infty}^{\infty} \frac{\rho}{(\rho^2 + u^2)^{3/2}} du \, \mathrm{a}_z\right]$$

$$* \int_{-\infty}^{\infty} \frac{\rho}{(\rho^2+u^2)^{3/2}} du \, \mathrm{a}_\rho$$

$$Let \;\; u = \rho \, \tan x \qquad , du = \rho sec^2 x \, dx \qquad x = \tan^{-1}\frac{u}{\rho}$$

$$x_1 = \tan^{-1}\infty = \frac{\pi}{2} \qquad , \quad x_2 = \tan^{-1}(-\infty) = \frac{-\pi}{2}$$

$$\int_{\infty}^{-\infty} \frac{\rho}{(\rho^2+u^2)^{3/2}} du = \int_{\frac{\pi}{2}}^{\frac{-\pi}{2}} \frac{\rho}{(\rho^2+\rho^2\tan^2 x)^{3/2}} \rho \, sec^2 x \, dx = \int_{\frac{\pi}{2}}^{\frac{-\pi}{2}} \frac{\rho}{\left(\rho^2(1+tan^2 x)\right)^{3/2}} sec^2 x \, dx$$

$$= \int_{\frac{\pi}{2}}^{\frac{-\pi}{2}} \frac{\rho}{(\rho^2 sec^2 x)^{3/2}} sec^2 x \, dx = \int_{\frac{\pi}{2}}^{\frac{-\pi}{2}} \frac{\rho^2}{\rho^3 sec^3 x} sec^2 x \, dx = \int_{\frac{\pi}{2}}^{\frac{-\pi}{2}} \frac{dx}{\rho \sec x}$$

$$= \frac{1}{\rho}\int_{\frac{\pi}{2}}^{\frac{-\pi}{2}} \cos x \, dx = \frac{1}{\rho}[sinx]_{\frac{\pi}{2}}^{\frac{-\pi}{2}} = \frac{1}{\rho}(-1-1) = \frac{-2}{\rho}\,\mathrm{a}_\rho$$

$$* \int_{\infty}^{-\infty} \frac{u}{(\rho^2+u^2)^{3/2}} du = \int_{\infty}^{-\infty} (\rho^2+u^2)^{-3/2} \; u \, du = \frac{1}{2}\int_{\infty}^{-\infty} (\rho^2+u^2)^{\frac{-3}{2}} 2u \;\, du$$

$$= \frac{1}{2}\left[\frac{(\rho^2+u^2)^{\frac{-1}{2}}}{\frac{-1}{2}}\right]_{\infty}^{-\infty} = \frac{1}{2}\times 2\left[\frac{-1}{\sqrt{\rho^2+u^2}}\right]_{\infty}^{-\infty} = \frac{-1}{\sqrt{\infty}} + \frac{1}{\sqrt{\infty}} = 0+0 = 0$$

$$E = \frac{-\rho_L}{4\pi\varepsilon_0}\left[\frac{-2}{\rho}\,\mathrm{a}_\rho + 0\right] = \frac{\rho_L}{4\pi\varepsilon_0\rho}\,\mathrm{a}_\rho$$

$\vec{E} = \dfrac{\rho_L}{2\pi\varepsilon_o\rho}\mathbf{a}_\rho$ in cylindrical or $\vec{E} = \dfrac{\rho_L}{2\pi\varepsilon_o}\dfrac{x\mathrm{a}_x + y\mathrm{a}_y}{(x^2+y^2)}$ in Cartesian

if a line charge is located at (x_0, y_0) $\quad E = \dfrac{\rho_L}{2\pi\varepsilon_o}\dfrac{(x-x_0)\mathrm{a}_x + (y-y_0)\mathrm{a}_y}{((x-x_0)^2+(y-y_0)^2)}$

Example: A uniform line charge of 16nC/m is located along the z-axis: (a) find E at P(1, 2, 3) (b) Q(1, 2, 7) ?

Solution:

(a)

$$E = \frac{\rho_L}{4\pi\varepsilon_0\rho}\,\mathrm{a}_\rho$$

$$\rho = \sqrt{x^2 + y^2} = \sqrt{1^2 + 2^2} = \sqrt{5}$$

$$E = \frac{16 \times 10^{-9}}{4\pi \times 8.85 \times 10^{-12} \times \sqrt{5}}\,\mathrm{a}_\rho = 128.62\,\mathrm{a}_\rho\ V/m$$

$$\text{in Cartesian } E = \frac{\rho_L}{2\pi\varepsilon_o}\frac{x\mathrm{a}_x + y\mathrm{a}_y}{(x^2 + y^2)} = \frac{16 \times 10^{-9}}{2\pi \times 8.85 \times 10^{-12}}\frac{\mathrm{a}_x + 2\mathrm{a}_y}{(1^2 + 2^2)} = 57.54\mathrm{a}_x + 115\mathrm{a}_y$$

$$(\boldsymbol{b})\quad \rho = \sqrt{x^2 + y^2} = \sqrt{1^2 + 2^2} = \sqrt{5}$$

$$E = \frac{16 \times 10^{-9}}{4\pi \times 8.85 \times 10^{-12} \times \sqrt{5}}\,\mathrm{a}_\rho = 128.62\,\mathrm{a}_\rho\ V/m$$

Example: Two uniform line charges of $\rho_L = 5$nC/m each are parallel to the x axis, one at z = 0, *y* = -*2* m and the other at z = 0, y = 4m. Find **E** at (4, 1, 3) m?

Solution:

$$E = E_1 + E_2$$

$$E_1 = \frac{\rho_L}{2\pi\varepsilon_o}\frac{(y - y_0)\mathrm{a}_y + (z - z_0)\mathrm{a}_z}{((y - y_0)^2 + (z - z_0)^2)}$$

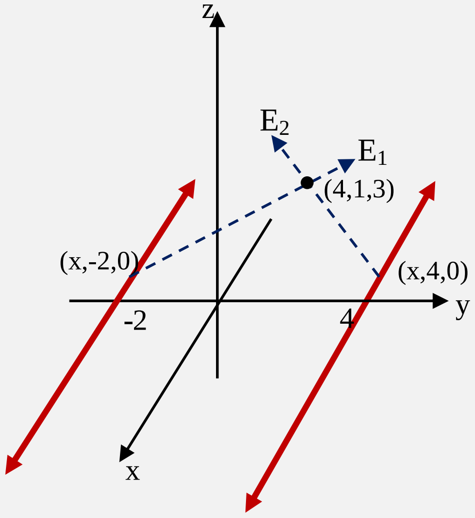
$$E_1 = \frac{5 \times 10^{-9}}{2\pi \times 8.85 \times 10^{-12}}\frac{\big(1 - (-2)\big)\mathrm{a}_y + (3 - 0)\mathrm{a}_z}{(1 - (-2)^2 + (3 - 0)^2)}$$

$$E_1 = \frac{5 \times 10^{-9}}{2\pi \times 8.85 \times 10^{-12}}\frac{3\mathrm{a}_y + 3\mathrm{a}_z}{18}$$

$$E_2 = \frac{5 \times 10^{-9}}{2\pi \times 8.85 \times 10^{-12}}\frac{(1 - 4)\mathrm{a}_y + (3 - 0)\mathrm{a}_z}{((1 - 4)^2 + (3 - 0)^2)}$$

$$E_1 = \frac{5 \times 10^{-9}}{2\pi \times 8.85 \times 10^{-12}}\frac{-3\mathrm{a}_y + 3\mathrm{a}_z}{18}$$

$$E = E_1 + E_2 = 2 * \frac{5 \times 10^{-9}}{2\pi \times 8.85 \times 10^{-12}}\frac{3\mathrm{a}_z}{18} = 30\mathrm{a}_z$$

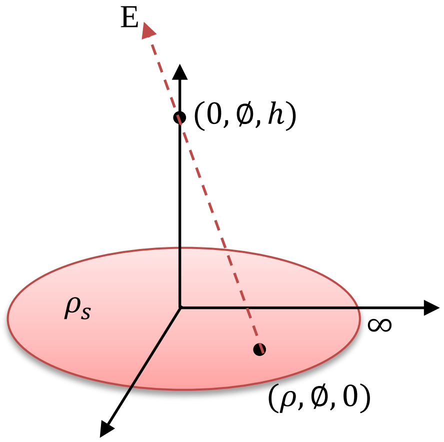


6. ***Field of a Sheet Charge***

Another basic charge configuration is the infinite sheet of charge having a uniform density of ρ_S C/m^2. Such a charge distribution may often be used to approximate that found on the conductors of a strip transmission line or a parallel-plate capacitor. Where static charge resides on conductor surfaces and not in their interiors; for this reason, ρ_S is commonly known as *surface charge density.*

اللوح الانهائي للشحنة، ذو الكثافة المنتظمة ρ_S C/m^2 ، هو شكل اساسي اخر للشحنات . ومثل هذا التوزيع للشحنة قد يستخدم كثيرا لتقريب ذلك الموجود على المكثف ذي اللوحين المتوازيين . حيث ان الشحنة الاستاتيكية تستوطن اسطح الموصلات وليس في داخلها ولهذا السبب فان ρ_S تعرف عامة بكثافة الشحنة السطحية.

Consider an ***infinite sheet charge*** in the xy-plane with uniform density ρ_S. The charge associated with elemental area *dS* is



$$Q = \int \rho_S \, ds$$

$$\vec{E} = \int \frac{\rho_S \, ds}{4\pi\varepsilon_o R^2} \mathbf{a}_R$$

$$ds = \rho \, d\rho d\emptyset$$

$$R = -\rho \mathrm{a}_\rho + h\mathrm{a}_z \qquad , \qquad |R| = \sqrt{\rho^2 + h^2}$$

$$E = \int_0^{2\pi} \int_{\rho=0}^{\infty} \frac{\rho_s \, \rho d\rho d\emptyset}{4\pi\varepsilon_0 \left(\sqrt{\rho^2 + h^2}\right)^2} \frac{-\rho \mathrm{a}_\rho + h\mathrm{a}_z}{\sqrt{\rho^2 + \mathrm{h}^2}}$$

$$E = \frac{h\rho_s}{4\pi\varepsilon_0} \int_0^{2\pi} \int_{\rho=0}^{\infty} \frac{\rho d\rho d\emptyset}{(\rho^2 + h^2)^{\frac{3}{2}}} \mathrm{a}_z$$

$$= 2\pi * \frac{h\rho_s}{4\pi\varepsilon_0} \int_{\rho=0}^{\infty} \frac{\rho d\rho}{(\rho^2 + h^2)^{\frac{3}{2}}} \mathrm{a}_z$$

$$= \frac{h\rho_s}{2\varepsilon_0} \int_{\rho=0}^{\infty} \rho(\rho^2 + h^2)^{\frac{-3}{2}} d\rho \, \mathrm{a}_z$$

$$\frac{h\rho_s}{2\varepsilon_0}\frac{1}{2}\left[\frac{-2}{\sqrt{\rho^2+h^2}}\right]_0^{\infty} \mathrm{a}_z$$

$$= \frac{h\rho_s}{4\varepsilon_0}\left[\frac{-2}{\infty}+\frac{2}{\sqrt{0+h^2}}\right]\mathrm{a}_z$$

$$= \frac{h\rho_s}{4\varepsilon_0}\frac{2}{h}\mathrm{a}_z = \frac{\rho_s}{2\varepsilon_0}\mathrm{a}_z$$

$$\vec{E} = \frac{\rho_S}{2\varepsilon_o}\mathrm{a}_N$$

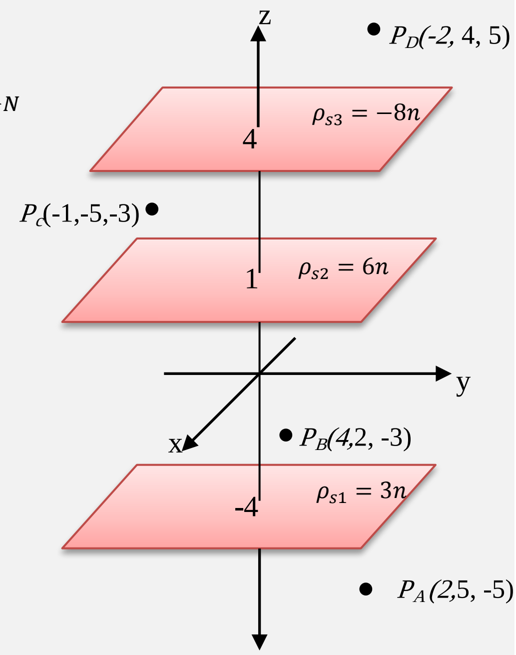
Where a_N is a unit vector normal to the sheet

Example: Three infinite uniform sheets of charge are located in free space as follows: 3nC/ m^2 at z = -4, 6nC/m^2 at z = 1, and - 8nC/m^2 at z = 4. Find E at the point: (a) $P_A(2, 5, -5)$; (b) $P_B(4, 2, -3)$; (c) $P_c(-1, -5, 2)$; (d) $P_D(-2, 4, 5)$?

Solution:

$a -\quad at\ p_A$

$$Becuse\ the\ infinite\ sheet\ charge\ the\ E = \vec{E} = \frac{\rho_S}{2\varepsilon_o}\mathrm{a}_N$$



$$E_T = \left[\frac{-3n}{2\varepsilon_0} - \frac{6n}{2\varepsilon_0} + \frac{8n}{2\varepsilon_0}\right]\mathrm{a}_z = -56.5\ \mathrm{a}_z\ V/m$$

b- $\quad at\ p_B$

$$E = \left[\frac{3n}{2\varepsilon_0} - \frac{6n}{2\varepsilon_0} + \frac{8n}{2\varepsilon_0}\right]\mathrm{a}_z = 282.3\ \mathrm{a}_z\ V/m$$

c- $\quad at\ p_C$

$$E = \left[\frac{3n}{2\varepsilon_0} + \frac{6n}{2\varepsilon_0} + \frac{8n}{2\varepsilon_0}\right]\mathrm{a}_z = 960.45\mathrm{a}_z\ V/m$$

$d -\quad at\ p_d$

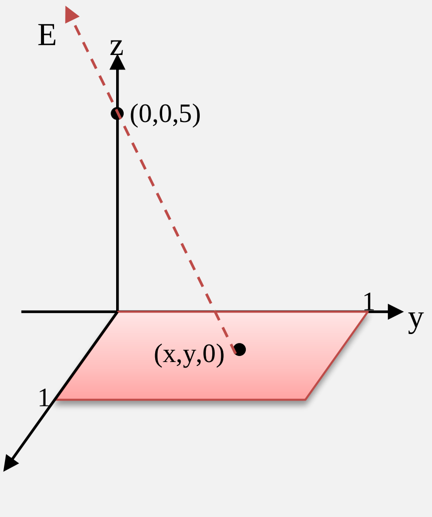
$$E = \left[\frac{3n}{2\varepsilon_0} + \frac{6n}{2\varepsilon_0} - \frac{8n}{2\varepsilon_0}\right]\mathrm{a}_z = 56.5\mathrm{a}_z\ V/m$$

Example: The finite sheet $0 \le x \le 1, 0 \le y \le 1$ on the z=0 plane has a charge density $\rho_S = xy(x^2+y^2+25)^{\frac{3}{2}}$ nC/m^2. Find:

a- The electric field (E) at (0, 0, 5)?

b- The force experienced by a -1 mC charge located at (0, 0, 5)?

Solution:



$$\vec{E} = \int \frac{\rho_S\, ds}{4\pi\varepsilon_o R^2}\mathbf{a}_R$$

$$ds = dxdy$$

$$R = -x\mathrm{a}_x - y\mathrm{a}_y + 5\mathrm{a}_z$$

$$|R| = \sqrt{x^2+y^2+25}$$

$$E = \int_0^1\int_0^1 \frac{xy(x^2+y^2+25)^{\frac{3}{2}}\,dxdy}{4\pi\varepsilon_0\left(\sqrt{x^2+y^2+25}\right)^2}\,\frac{-x\mathrm{a}_x - y\mathrm{a}_y + 5\mathrm{a}_z}{\sqrt{x^2+y^2+25}}$$

$$E = \frac{1\times10^{-9}}{4\pi\varepsilon_0}\int_0^1\int_0^1 \frac{xy(x^2+y^2+25)^{\frac{3}{2}}\,dxdy}{(x^2+y^2+25)^{\frac{3}{2}}}\left(-x\mathrm{a}_x - y\mathrm{a}_y + 5\mathrm{a}_z\right)$$

$$E = \frac{1\times10^{-9}}{4\pi\varepsilon_0}\int_0^1\int_0^1 xydxdy\left(-x\mathrm{a}_x - y\mathrm{a}_y + 5\mathrm{a}_z\right)$$

$$E = \frac{1\times10^{-9}}{4\pi\varepsilon_0}\left[-\int_0^1\int_0^1 x^2ydxdy\,\mathrm{a}_x - \int_0^1\int_0^1 xy^2dxdy\,\mathrm{a}_y + 5\int_0^1\int_0^1 xydxdy\,\mathrm{a}_z\right]$$

$$E = \frac{1\times10^{-9}}{4\pi\varepsilon_0}\left[-\left[\frac{x^3}{3}\right]_0^1\left[\frac{y^2}{2}\right]_0^1\mathrm{a}_x - \left[\frac{x^2}{2}\right]_0^1\left[\frac{y^3}{3}\right]_0^1\mathrm{a}_y + 5\left[\frac{x^2}{2}\right]_0^1\left[\frac{y^2}{2}\right]_0^1\mathrm{a}_z\right]$$

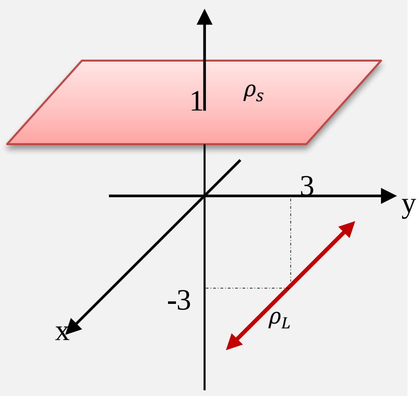
$$E = \frac{1\times10^{-9}}{4\pi\varepsilon_0}\left[-\frac{1}{3}\frac{1}{2}\mathrm{a}_x - \frac{1}{2}\frac{1}{3}\mathrm{a}_y + 5\frac{1}{2}\frac{1}{2}\mathrm{a}_z\right] = \frac{1\times10^{-9}}{4\pi\varepsilon_0}\left[-\frac{1}{6}\mathrm{a}_x - \frac{1}{6}\mathrm{a}_y + \frac{5}{4}\mathrm{a}_z\right]$$

$$E = -1.5\mathrm{a}_x - 1.5\mathrm{a}_y + 11.23\mathrm{a}_z$$

$$(b)\ F = QE = -1\times10^{-9}\left(-1.5\mathrm{a}_x - 1.5\mathrm{a}_y + 11.23\mathrm{a}_z\right)$$

Example: A uniform sheet charge with $\rho_s = 1/3\pi$ nC/m^2 is located at z = 5 m and a uniform line charge with $\rho_L = -25/9$ nC/m at z =-3 m, y = 3m. Find E at (x, -1, 0)m.?

Solution:



$E = E_1 + E_2$

E_1 due to a surface charge

E_2 due to a line charge

$$E_1 = \frac{\rho_s}{2\varepsilon_0} a_N$$

$$E_1 = \frac{(1/3\pi)\times 10^{-9}}{2\varepsilon_0}(-a_z) = -6a_z$$

$$E_2 = \frac{\rho_L}{2\pi\varepsilon_o}\frac{(y-y_0)a_y + (z-z_0)a_z}{((y-y_0)^2 + (z-z_0)^2)}$$

$$E_2 = \frac{(-25/9)\times 10^{-9}}{2\pi\varepsilon_o}\frac{(-1-3)a_y + (0-(-3))a_z}{((-1-3)^2 + (0-(-3))^2)}$$

$$E_2 = \frac{(-25/9)\times 10^{-9}}{2\pi\varepsilon_o}\frac{-4a_y + 3a_z}{16+9}$$

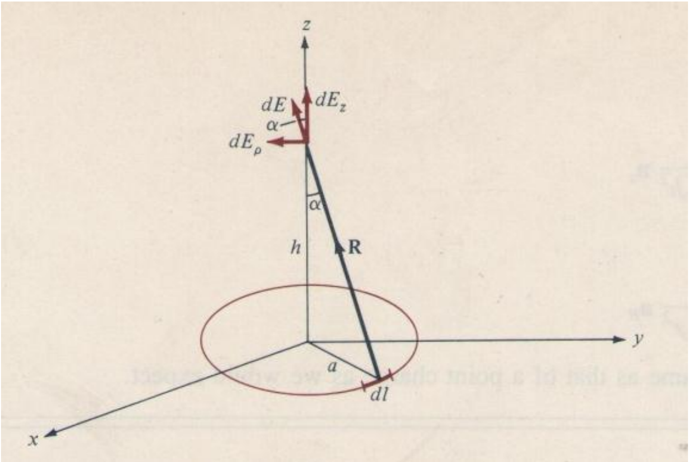
$$E_2 = 8a_y - 6a_z$$

$$E_T = -6a_z + 8a_y - 6a_z = 8a_y - 12a_z$$

Home work

Q₁: Let a point charge Q_1 = 25 nC be located at P_1(4, -2, 7) and a charge Q_2 = 60 nC be at P2(-3, 4, -2). (a) Find E at P(1, 2, 3). (b) At what point on the y-axis is $E_x = 0$?

Q₂: A circular ring of radius ***a*** carries a charge Q C and is placed on the xy-plane, find electric field E at (0, 0, h)



Q₃: The circular disk $r \leq 2m$ in the z = 0 plane has a charge density $\rho_s = \frac{10^{-8}}{r}$.Determine the electric field E at the point (0, 0, *h)?*

Q₄: Two infinite sheets of uniform charge density $\rho_s = 10^{-9}/6\pi$ C/m^2 are located at z = -5 m and y = - 5 m. Determine the uniform line charge density p_L necessary to produce the same value of E at (4,2,2) m, if the line charge is located at z = 0, y = 0.

Ans. **0.667 nC/m**

Q₅: A uniform line charge of $\rho_L = \sqrt{2} \times \frac{10^{-8}}{6}$ C/m lies along the *x* axis and a uniform sheet of charge is located at y = 5 m. Along the line *y* = 3 m, z = 3 m the electric field E has only $\mathbf{a}_z$ component. What is ρ_S for the sheet?

Ans. **125 pC/m²**